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Zeta regularized determinant of the Laplacian for classes of spherical space forms

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Abstract

We derive the spectral zeta function in terms of certain Dirichlet series for a variety of spherical space forms M_G. Extending results in [C. Nash, D. O'Connor, Determinants of Laplacians on lens spaces, J. Math. Phys. 36 (3) (1995) 1462–1505] the zeta-regularized determinant of the Laplacian on M_G is obtained explicitly from these formulas. In particular, our method applies to manifolds of dimension higher than 3 and it includes the case where G arises from the dihedral group of order $2m$. As a crucial ingredient in our analysis we determine the dimension of eigenspaces of the Laplacian in form of some combinatorial quantities for various infinite classes of manifolds from the explicit form of the generating function in [A. Ikeda, On the spectrum of a Riemannian manifold of positive constant curvature, Osaka J. Math. 17 (1980) 75–93].

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1. Introduction

This paper is a continuation of a previous work, [6] which deals with the Laplacian on Heisenberg manifolds. Therein we gave an expression of the zeta-regularized determinant of the Laplacian for three and five dimensional Heisenberg manifolds. In these cases, the spectral zeta function is a restriction of a certain kind of a multiple zeta function given in the form

$$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{1}{n^{s-1}(n+\alpha m+\beta)^s}, \tag{1.1}$$

or

$$\sum_{n=1}^{\infty}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{1}{n^{s-2}(n+\alpha m_1+\beta m_2+\gamma)^s}, \tag{1.2}$$

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according to the dimension of the Heisenberg manifold ($\alpha, \beta, \gamma > 0$). It is known that these functions can be continued meromorphically to the complex plane. In order to derive the zeta-regularized determinant, explicit formulas for (1.1) or (1.2) valid in a domain containing the value $s = 0$ have been applied and can be obtained via iterated use of partial integrations for an integral expression of the above functions.

In the present paper we calculate the spectral zeta function ζ_{M_G} as well as the zeta-regularized determinant of the Laplacian for certain classes of spherical space forms and lens spaces $M_G := \mathbb{S}^{2N+1}/G$. Here $G \subset SO(2N+2)$ is a finite subgroup acting freely on $\mathbb{S}^{2N+1}$ (cf. Theorems 4.1–4.3). In particular, our analysis leads to an alternative proof for the case of three dimensional lens spaces which was treated before in [13]. Moreover, it extends to cases of higher dimensions and our methods also apply if G arises from the dihedral group. We show that all spectral zeta functions appearing in this paper can be expressed as a combination of the Dirichlet series

$$\sum_{k=0}^{\infty} \frac{1}{(k+\alpha)^{s-d}(k+\beta)^{s}}, \quad \alpha, \beta > 0 \tag{1.3}$$

where d is related to the dimension of $\mathbb{S}^{2N+1}/G$ and the group G. The possible distinct eigenvalues of the Laplacian on spherical space forms are known. Compared to the analysis of (1.1) and (1.2), it is rather easy to handle the analytic continuation of (1.3) at the value $s = 0$ and to calculate all poles and corresponding residues. For this purpose we express (1.3) in an integral form with congruent hypergeometric functions (cf. [12]) and employ the so called *Egami interpolation method* (cf. [5]).

However, to determine ζ_{M_G}, we must calculate the multiplicities of all distinct eigenvalues of the Laplacian on M_G. In the paper [8], it was proved that the generating function of the multiplicities of the Laplacian on spherical space forms is a rational function in an explicit form. By making use of this fact we determine the multiplicity of each eigenvalue. In the case of three dimensional lens spaces, and different from our method, such multiplicities have been derived in [13]. Therein the authors use expressions of the characters of the isometry group G (where G is a cyclic group) acting on each eigenspace of the Laplacian on the three dimensional sphere $\mathbb{S}^3$.

Let P be a first order positive elliptic pseudo-differential operator defined on a closed manifold with vanishing sub-principal symbol and periodic bicharacteristic flow (of a common period). It was proved in [3] that except a finite number of eigenvalues, the multiplicities of the distinct eigenvalues of P are given by a polynomial of order one less than the dimension of the manifold according to the numbering of the distinct eigenvalues. In case of the spherical space forms treated in the present paper, geodesics are always periodic and so the bicharacteristic flow of (the square root of) the Laplacian is periodic. Here we show that the corresponding spectral zeta function is decomposed into a finite number of Dirichlet series (1.3) and in each series the coefficients are expressed by a polynomial (cf. Corollaries 2.1 and 2.2). Such a decomposition enables us to calculate the zeta-regularized determinant for spherical space forms.

As is well-known, *Kronecker's second limit formula* for two dimensional tori is one of the most interesting formulas involving zeta-regularized determinants. Many proofs of it are known (cf. [4,6,7,10] and etc.) and one typical method is based on a functional equation of the *Epstein zeta function*. Such a kind of equation serves to obtain the derivative of a function at the origin by evaluating it at a certain point. Although it is not possible to apply these ideas directly in our approach, in [9] Matsumoto has presented a candidate of a functional equation for a class of multiple zeta functions that includes expressions of the form (1.1). A similar functional equation can be expected to hold for the Dirichlet series (1.3) (cf. [12]).

In Section 2 we discuss the spectral structure of the Laplacian Δ_{M_G} on general spherical space forms. There it is shown that the eigenvalues of Δ_{M_G} are divided into a finite series and in each such series the coefficients are expressed by a polynomial. Moreover, the spectral zeta function ζ_{M_G} of M_G is decomposed into a sum of a meromorphic function $\zeta_{\mathbb{S}^{2N+1}}/|G|$ and an entire function h_G. We express both, the function h_G as well as the zeta-regularized determinant of the Laplacian on M_G in terms of the spectral data (cf. Proposition 2.1 and Corollary 2.1).

Section 3 serves to calculate the meromorphic extension of the Dirichlet series (1.3) for all $d = 0, 1, 2 \ldots$. We characterize its poles and calculate all residues which are polynomials in α and β (cf. Proposition 3.1). Furthermore its values and derivatives at the point $s = 0$ are obtained in terms of the *Hurwitz zeta function* and its derivatives. Our calculations are based on *Egami's interpolation method*.

In Section 4 we combine the formulas given in Sections 2 and 3 to calculate the spectral zeta function and the zeta-regularized determinant more explicitly for several classes of lens spaces and spherical space forms. Among our examples we also deal with cases of (odd) dimension higher than three and manifolds different from the lens spaces. Appendix A at the end of this paper provides some frequently used relations between the Hurwitz zeta function and Bernoulli polynomials as well as all the combinatorial quantities that appear in the formulas of Section 4.

Finally, in Appendix B and by using a power series expansion we explain an alternative method for expressing the analytic continuation of the Dirichlet series (1.3) in a domain including $s = 0$ for $d = 0$ and $d = 1$. Different methods will lead to different expressions of the resulting quantities. This way was employed before in the paper [17] for calculating the zeta-regularized determinant of spheres. Therein the author gave a multiplicative form in terms of *Barne's multiple Gamma functions* $\{\Gamma_n(s)\}$ where $n \in \mathbb{N}_0$ (here $\Gamma_0(x) = x^{-1}$ and Γ_1 coincides with the usual Gamma function).

For $d = 0$ and by making use of *Weierstrass's canonical form* of the Gamma function we remark, that the two different expressions of the value $H'_{\alpha,\beta}(0, 0)$ (cf. (2.13) and (2.14)) which we have obtained in the present paper coincide. For $d = 1$ and by comparing two different formulas for the value $H'_{\alpha,\beta}(-1, 0)$ we obtain an expression of Barne's G-function ($=\Gamma_2(\alpha)$) in terms of the derivative $\zeta'(-1, \alpha)$ (=Hurwitz zeta function) and $\Gamma(\alpha)$. Such kinds of formula are part of more general relations between the Hurwitz zeta function and multiple Gamma functions as was pointed out in [2].

2. Spectral zeta function of spherical space forms

Let M be a compact and connected Riemannian manifold of dimension $\dim M \geq 2$ (without boundary) and with Laplace–Beltrami operator Δ_M acting on smooth functions. It is well-known, that the spectrum of Δ_M consists of only eigenvalues with finite multiplicities. Let

$$0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots$$

be the distinct eigenvalues and we denote the multiplicity of each eigenvalue by $\mathbf{m}_k(M)$ where $\mathbf{m}_0(M) = 1$. The *spectral zeta function* ζ_M of M is defined as

$$\zeta_M(s) = \sum_{k=1}^{\infty} \frac{\mathbf{m}_k(M)}{\lambda_k^s} = \mathrm{Trace}\left(\Delta_M^{-\frac{s}{2}}\right)$$

where $\mathrm{Re}(s) > \dim M/2$. As a basic property ζ_M can be continued as a meromorphic function to the complex plane with only simple poles at the points $s = \dim M/2 - k$ ($k \in \mathbb{N}_0$). It is known that this is equivalent to the fact that the heat kernel has an asymptotic expansion as $t \downarrow 0$ through the Mellin transformation (cf. [11]). In particular, the spectral zeta functions are holomorphic at the origin.

Throughout this paper we assume that M is a spherical space form, that is let $\mathbb{S}^N$ for $N \geq 2$ be the N-dimensional sphere in $\mathbb{R}^{N+1}$ and fix a *finite group G of fix point free isometries* on $\mathbb{S}^N$ such that M is isometric to $\mathbb{S}^N/G$. According to the classification result in [14,19] the even dimensional spherical space forms are only the spheres and the real projective spaces. In the following we deal with the odd dimensional cases and set

$$M_G := \mathbb{S}^{2N+1}/G, \quad N \in \mathbb{N}.$$

We remark (cf. [8]), that finite fix point free subgroups of isometries on $\mathbb{S}^{2N+1}$ are contained in the special orthogonal group $SO(2N + 2)$ and $\lambda = 1$ is not an eigenvalue of each $g \neq I$ in G (I = identity matrix). The (distinct) eigenvalues of $\Delta_{\mathbb{S}^{2N+1}}$ and Δ_{M_G} coincide and they are given by:

$$\mathbf{E}_{2N+1} = \{\lambda_k := k(k + 2N) : k = 0, 1, 2, \ldots\}$$

with multiplicities $\mathbf{m}_k := \mathbf{m}_k(\mathbb{S}^{2N+1})$ and $\mathbf{m}_k(M_G)$, respectively. It is known that

$$\mathbf{m}_k = \frac{2(N + k)}{(2N)!} \prod_{l=1}^{2N-1} (k + l) \tag{2.1}$$

(cf. Lemma 4.1) and $\mathbf{m}_k(M_G)$ coincides with the dimension of the G-invariant eigenfunctions in the eigenspace of $\Delta_{\mathbb{S}^{2N+1}}$ corresponding to λ_k. Therefore $\mathbf{m}_k(M_G)$ completely characterizes the spectrum of Δ_{M_G}.

Definition 2.1. The *generating function* associated with the spectrum of Δ_{M_G} is defined by:

$$F_G(z) := \sum_{k=0}^{\infty} \mathbf{m}_k(M_G) z^k. \tag{2.2}$$

F_G has a meromorphic continuation to the complex plane $\mathbb{C}$. In terms of a meromorphic extension of the spectral zeta-function ζ_{M_G} the *zeta-regularized* determinant of Δ_{M_G} is given by :

$$\det \Delta_{M_G} := \exp\left\{-\frac{\partial \zeta_{M_G}}{\partial s}(0)\right\}$$

(for the definition see [7,15–17]). As a crucial observation the generating function determines the spectral zeta function of a spherical space form completely and its meromorphic extension is rational (cf. [8]):

Theorem 2.1 (*Ikeda, [8]*). *On the domain* $\{z \in \mathbb{C} : |z| < 1\}$ *the series* (2.2) *converges to the function:*

$$F_G(z) = \frac{1}{|G|}\sum_{g\in G} \frac{1-z^2}{\det(I-gz)} = \frac{1-z^2}{|G|}\sum_{g\in G}\prod_{\gamma\in E(g)} \frac{1}{(z-\gamma)^{m_\gamma}},$$

where $E(g)$ *is the set of distinct eigenvalues of* g *and* m_γ *denotes the multiplicity of* γ.

Let $r > 1$, then from (2.2) and residue calculus it follows:

$$\mathbf{m}_k(M_G) = \frac{1}{2\pi\sqrt{-1}}\int_{|z|=r} \frac{F_G(z)}{z^{k+1}}\mathrm{d}z - \frac{1}{|G|}\sum_{g\in G}\sum_{\gamma\in E(g)} \operatorname{Res}\left(\frac{1-z^2}{z^{k+1}\det(I-gz)}, \gamma\right).$$

Since for $|z| = r > 1$, $k \geq 0$ and $N \geq 1$

$$\left|\frac{1-z^2}{z^{k+1}\det(I-gz)}\right| \leq \frac{r^2+1}{r^{k+1}(r-1)^{2N+2}} = O\left(r^{-2N-k-1}\right) \quad (r\to\infty)$$

one concludes that $\int_{|z|=r}\frac{F_G(z)}{z^{k+1}}\mathrm{d}z = 0$ and hence the spectral zeta-function of the spherical space form M_G can be rewritten as:

Proposition 2.1. *For* $\mathrm{Re}(s) > \frac{2N+1}{2}$ *there is a decomposition:*

$$\zeta_{M_G}(s) = \frac{\zeta_{\mathbb{S}^{2N+1}}(s)}{|G|} + h_G(s) \tag{2.3}$$

and h_G *extends to an entire function on the complex plane. Moreover:*

$$h_G(s) = \frac{1}{|G|}\sum_{g\in G\setminus\{I\}}\sum_{\gamma\in E(g)}\sum_{k=1}^{\infty}\frac{C_k(g,\gamma)}{\lambda_k^s}$$

where $C_k(g,\gamma) := \operatorname{Res}\left(\frac{z^2-1}{z^{k+1}\det(I-gz)}, \gamma\right)$. *The dimension of the eigenspaces of* Δ_{M_G} *are:*

$$\mathbf{m}_k(M_G) = \frac{1}{|G|}\sum_{g\in G}\sum_{\gamma\in E(g)} C_k(g,\gamma).$$

Proof. The decomposition (2.3) follows from our remarks above and $\mathbf{m}_k = C_k(I,1)$. Hence it is enough to show, that h_G has no poles. Through the Mellin transformation it is known that the integrals over the coefficients of the heat kernel asymptotics (which are functions on the sphere determined by metric tensors and their derivatives) coincide with the residues at the poles of the spectral zeta function. In the case of the sphere $\mathbb{S}^{2N+1}$ and the spherical space form M_G, these coefficients are constant and their integrals only differ by the factor $|G|$ (the group order). Therefore all poles and residues of ζ_{M_G} coincide with the poles and residues of the first term of the right hand side of (2.3). As a consequence the second term, h_G, must be an entire function. $\square$

In the following we put for $g \in G$ and $\gamma \in E(g)$:

$$J_{g,\gamma}(s) := \sum_{k=1}^{\infty} \frac{C_k(g,\gamma)}{\lambda_k^s}. \tag{2.4}$$

Corollary 2.1. *The zeta-regularized determinant of Δ_{M_G} is:*

$$\log\det\Delta_{M_G} = \frac{1}{|G|}\log\det\Delta_{\mathbb{S}^{2N+1}} - \frac{1}{|G|}\sum_{g\in G\setminus\{I\}}\sum_{\gamma\in E(g)} \frac{\partial J_{g,\gamma}}{\partial s}(0). \tag{2.5}$$

In order to calculate (2.5) we have to derive an analytic continuation of $J_{g,\gamma}(s)$ to a domain in $\mathbb{C}$ containing $s = 0$. Let $g \in G$ be of order $q_g > 2$. Since $I \neq g^2 \in G$ is fix point free $-1 \notin E(g)$. Moreover, each $\gamma \in E(g)$ is a q_g-th root of unity. We set $\delta := \exp(2\pi \mathrm{i}/q_g)$ and write

$$E(g) = \left\{\delta^{p_j}, \delta^{-p_j} : j = 0, \ldots r\right\}$$

where $\delta^{p_j}, \delta^{-p_j}$ have multiplicity i_j and $0 < p_0 < p_1 < \cdots < p_r < q_g/2$ are integers. It clearly holds that $i_0 + i_1 + \cdots + i_r = N + 1$ and we can write:

$$\frac{z^2-1}{\det(I-gz)} = \frac{z^2-1}{\prod_{j=0}^{r}(z-\delta^{p_j})^{i_j}(z-\delta^{-p_j})^{i_j}}. \tag{2.6}$$

Fix an eigenvalue $\tilde{\gamma} \in E(g)$ with multiplicity i_j. For $\rho \neq \tilde{\gamma}$ consider the Taylor expansion at $\tilde{\gamma}$:

$$\frac{1}{(z-\rho)^{k+1}} = \sum_{n=0}^{\infty}(-1)^n \binom{k+n}{n} (\tilde{\gamma}-\rho)^{-k-n-1}(z-\tilde{\gamma})^n. \tag{2.7}$$

By choosing $\rho := 0$ and using (2.6) there is a function F holomorphic in a neighborhood of $\tilde{\gamma}$ such that:

$$\frac{z^2-1}{z^{k+1}\det(I-gz)} = F(z)\sum_{n=0}^{\infty}(-1)^n \binom{k+n}{n} \tilde{\gamma}^{-k-n-1}(z-\tilde{\gamma})^{n-i_j}. \tag{2.8}$$

If we write $k = \ell q_g + \nu$ for $\nu = 0, \ldots, q_g - 1$ we conclude from $\tilde{\gamma}^{q_g} = 1$ and (2.8):

Corollary 2.2. *For $\nu \in \{0, \ldots, q_g - 1\}$ the residue:*

$$C_{\ell q_g+\nu}(g,\tilde{\gamma}) = \operatorname{Res}\left(\frac{z^2-1}{z^{k+1}\det(I-gz)}, \tilde{\gamma}\right) \tag{2.9}$$

is a polynomial in ℓ of (maximal) degree d with

$$d \leq \begin{cases} i_j - 1 \leq N & \text{if } g \notin \{-I, I\} \\ 2i_0 - 2 \leq 2N & \text{if } g \in \{-I, I\}. \end{cases}$$

In the following we write $P_{(g,\tilde{\gamma},\nu)}(\ell) := C_{\ell q_g+\nu}(g,\tilde{\gamma})$. Then for all $g \in G$ the function (2.4) can be decomposed as:

$$J_{g,\gamma}(s) = \frac{1}{q_g^{2s}}\sum_{\ell=1}^{\infty}\frac{P_{(g,\gamma,0)}(\ell)}{\ell^s\left(\ell+\frac{2N}{q_g}\right)^s} + \frac{1}{q_g^{2s}}\sum_{\nu=1}^{q_g-1}\sum_{\ell=0}^{\infty}\frac{P_{(g,\gamma,\nu)}(\ell)}{\left(\ell+\frac{\nu}{q_g}\right)^s\left(\ell+\frac{\nu+2N}{q_g}\right)^s}. \tag{2.10}$$

For $g \notin \{-I, I\}$ and $\nu = 0, \ldots, q_g - 1$ we set:

$$P_{(g,\gamma,\nu)}(\ell) = \sum_{r=0}^{N}\alpha_r(g,\gamma,\nu)\left(\ell+\frac{\nu}{q_g}\right)^r \tag{2.11}$$

where $\alpha_r(g, \gamma, \nu) \in \mathbb{C}$ are suitable coefficients. With $\alpha_r(g, \gamma, q_g) := \alpha_r(g, \gamma, 0)$ it can be checked from (2.10) that:

$$J_{g,\gamma}(s) = \frac{1}{q_g^{2s}} \sum_{r=0}^{N} \sum_{\nu=1}^{q_g} \alpha_r\,(g, \gamma, \nu)\, H_{\frac{\nu}{q_g}, \frac{\nu+2N}{q_g}}\,(s - r, s)\,. \tag{2.12}$$

Here for $\beta > \alpha > 0$ and $r \in \mathbb{N}_0$ we use the notation:

$$H_{\alpha,\beta}\,(s_1, s_2) := \sum_{\ell=0}^{\infty} \frac{1}{(\ell + \alpha)^{s_1} (\ell + \beta)^{s_2}} \tag{2.13}$$

which converges absolutely for $\mathrm{Re}(s_1 + s_2) > 1$. It is known (cf. [12], Theorem 1) that the *Dirichlet series* (2.13) can be continued meromorphically to all $s_1, s_2 \in \mathbb{C}$. Moreover, for all $r \in \mathbb{N}_0$ the assignment

$$\mathbb{C} \ni s \mapsto H_{\alpha,\beta}\,(s - r, s)$$

is holomorphic in a neighborhood of $s = 0$. In our notation we do not distinguish between the series (2.13) and its meromorphic extension. By writing:

$$H'_{\alpha,\beta}(-r, 0) := \frac{\partial}{\partial s} H_{\alpha,\beta}(s - r, s)_{|s=0} \tag{2.14}$$

we find from (2.12) and Corollary 2.1:

Proposition 2.2. *For $g \in G \setminus \{-I, I\}$ and $\gamma \in E(g)$:*

$$\frac{\partial}{\partial s} J_{g,\gamma}(0) = \sum_{r=0}^{N} \sum_{\nu=1}^{q_g} \alpha_r\,(g, \gamma, \nu) \left[H'_{\frac{\nu}{q_g}, \frac{\nu+2N}{q_g}}(-r, 0) - 2H_{\frac{\nu}{q_g}, \frac{\nu+2N}{q_g}}(-r, 0) \log q_g \right].$$

Moreover, with $\alpha_r\,(g, \nu) := \sum_{\gamma \in E(g)} \alpha_r\,(g, \gamma, \nu)$ one has:

$$|G| \log\det \Delta_{M_G} = D_L - \sum_{g \in G\setminus\{-I,I\}} \sum_{r=0}^{N} \sum_{\nu=1}^{q_g} \alpha_r\,(g, \nu) \left[H'_{\frac{\nu}{q_g}, \frac{\nu+2N}{q_g}}(-r, 0) - 2H_{\frac{\nu}{q_g}, \frac{\nu+2N}{q_g}}(-r, 0) \log q_g \right].$$

Here $D_L \in \mathbb{R}$ is given by

$$D_L := \begin{cases} 2 \log\det \Delta_{\mathbb{R}P^{2N+1}} & \textit{if } -I \in G \\ \log\det \Delta_{\mathbb{S}^{2N+1}} & \textit{else.} \end{cases}$$

Below we calculate $H'_{\alpha,\beta}(-r, 0)$ and $H_{\alpha,\beta}(-r, 0)$ for $\beta > \alpha > 0$ and all $r \in \mathbb{N}_0$ in terms of the *Hurwitz zeta function* and its derivative at certain non-positive integers. Moreover, for several examples of special groups G we derive a more explicit form of $\det \Delta_{M_G}$.

3. Meromorphic extension of a Dirichlet series

For $\beta > \alpha > 0$ and $\mathrm{Re}(s_1 + s_2) > 1$ let the *Dirichlet series* $H_{\alpha,\beta}(s_1, s_2)$ be defined as in (2.13). According to [12] and for $\mathrm{Re}(s) > d$ where $d \in \mathbb{N}_0$ there is an integral representation:

$$H_{\alpha,\beta}\,(s - d, s) = \frac{1}{(\beta - \alpha)^{2s-d-1}} \frac{\Gamma(2s - d)}{\Gamma(s - d)\Gamma(s)} \int_0^{\beta-\alpha} (\beta - \alpha - x)^{s-d-1} x^{s-1} \zeta\,(2s - d, x + \alpha)\, \mathrm{d}x \tag{3.1}$$

for $2s - d \neq 1$. Here $\zeta(s, x)$ denotes the *Hurwitz zeta function* defined by:

$$\zeta(s, x) := \sum_{n=0}^{\infty} \frac{1}{(n + x)^s}.$$

Let $k \in \mathbb{N}$ and $\mathrm{Re}(a) > 0$. Whenever defined we use the notation:

$$\zeta_x^{(k)}(s,a) := \frac{\partial^k \zeta}{\partial x^k}(s,x)_{|x=a}.$$

Moreover, we write $\zeta(s) := \zeta(s,1)$ for the usual *Riemann zeta function*. In (3.1) we use a decomposition called *Egami's interpolation method* (cf. [4,5]). Let $p, q \in \mathbb{N}$, then we write

$$\zeta(2s-d, x+\alpha) = P_{\alpha,\beta}^{p,q}(s,x) + F_{\alpha,\beta}^{p,q}(s,x) \tag{3.2}$$

where $P_{\alpha,\beta}^{p,q}$ is a polynomial in the variable x

$$P_{\alpha,\beta}^{p,q}(s,x) := (x-\beta+\alpha)^p A_{\alpha,\beta}^{p,q}(s,x) + x^q B_{\alpha,\beta}^{p,q}(s,x). \tag{3.3}$$

Here the expressions $A_{\alpha,\beta}^{p,q}$ and $B_{\alpha,\beta}^{p,q}$ denote suitable functions holomorphic in a zero neighborhood of $\mathbb{C}^2$. Moreover, the function $F_{\alpha,\beta}^{p,q}$ should be written as:

$$F_{\alpha,\beta}^{p,q}(s,x) := x^q (x-\beta+\alpha)^p G_{\alpha,\beta}^{p,q}(s,x) \tag{3.4}$$

where $G_{\alpha,\beta}^{p,q}$ is holomorphic in

$$\left\{ (s,x) \in \mathbb{C}^2 : s \neq \frac{d+1}{2}, \mathrm{Re}(x) > -\alpha \right\}.$$

It is easy to check that $A_{\alpha,\beta}^{p,q}$ is obtained as the Taylor polynomial in $x=0$ on the right hand side of:

$$\frac{\zeta(2s-d, x+\alpha)}{(x-\beta+\alpha)^p} = \underbrace{\sum_{m=0}^{q-1} a_m^{\alpha,\beta,p}(s) x^m}_{=:A_{\alpha,\beta}^{p,q}(s,x)} + O\left\{x^q\right\}$$

as $x \to 0$. The Taylor coefficients $a_m^{\alpha,\beta,p}(s)$ for $m = 0, \ldots, q-1$ are given by:

$$a_m^{\alpha,\beta,p}(s) = \sum_{l=0}^{m} (-1)^l \binom{p-1+l}{l} \frac{\zeta_x^{(m-l)}(2s-d,\alpha)}{(m-l)!\,(\alpha-\beta)^{p+l}}. \tag{3.5}$$

Using Lemma A.1, (3) and considering all empty products as one, it follows whenever the right hand side is defined:

$$a_m^{\alpha,\beta,p}(s) = (-1)^m \sum_{l=0}^{m} \binom{p-1+l}{l} \frac{\zeta(2s-d+m-l,\alpha)}{(m-l)!(\alpha-\beta)^{p+l}} \prod_{j=0}^{m-l-1}(2s-d+j). \tag{3.6}$$

Similarly, the function $B_{\alpha,\beta}^{p,q}$ is given by the Taylor polynomial in $x_0 := \beta - \alpha$ on the right hand side of:

$$\frac{\zeta(2s-d, x+\alpha)}{x^q} = \underbrace{\sum_{m=0}^{p-1} b_m^{\alpha,\beta,q}(s)(x-x_0)^m}_{=:B_{\alpha,\beta}^{p,q}(s,x)} + O\left\{(x-x_0)^p\right\}$$

as $x \to x_0$. For $2s-d \neq 1$ and $m = 0, \ldots, p-1$:

$$b_m^{\alpha,\beta,q}(s) = (-1)^m \sum_{l=0}^{m} \binom{q-1+l}{l} \frac{\zeta(2s-d+m-l,\beta)}{(m-l)!(\beta-\alpha)^{q+l}} \prod_{j=0}^{m-l-1}(2s-d+j). \tag{3.7}$$

Finally, a function $F_{\alpha,\beta}^{p,q}(s,x)$ having the desired properties is defined via (3.2). Note, that the integral

$$\int_0^{\beta-\alpha} (\beta-\alpha-x)^{s-d-1} x^{s-1} F_{\alpha,\beta}^{p,q}(s,x)\mathrm{d}x = (-1)^p \int_0^{\beta-\alpha} (\beta-\alpha-x)^{s+p-d-1} x^{s+q-1} G_{\alpha,\beta}^{p,q}(s,x)\mathrm{d}x \tag{3.8}$$

vanishes for $s = \frac{n}{2}$ whenever n is an integer with

$$\max\{2d + 2 - 2p, 2 - 2q, d - p - q + 2\} \leq n \leq d. \tag{3.9}$$

In fact, this is a consequence of (3.8) and:

Lemma 3.1. *Let $n \in \mathbb{Z}$ with $d - p - q + 2 \leq n \leq d$, then $G^{p,q}_{\alpha,\beta}(n/2, \cdot)$ and $F^{p,q}_{\alpha,\beta}(n/2, \cdot)$ vanish identically.*

Proof. For $d - p - q + 2 \leq n \leq d$ and by Lemma A.1, (1) it follows that:

$$F^{p,q}_{\alpha,\beta}(n/2, x) = -\frac{B_{d-n+1}(x+\alpha)}{d-n+1} - P^{p,q}_{\alpha,\beta}(n/2, x).$$

In particular, $F^{p,q}_{\alpha,\beta}(n/2, \cdot)$ is a polynomial of maximal degree $p + q - 1$. Since $G^{p,q}_{\alpha,\beta}(n/2, x)$ is holomorphic for $\mathrm{Re}(x) > -\alpha$ it follows from (3.4) that $F^{p,q}_{\alpha,\beta}(n/2, \cdot)$ has $p + q$ roots and vanishes identically. Therefore $G^{p,q}_{\alpha,\beta}(n/2, \cdot)$ vanishes as well. $\square$

3.1. The poles of $H_{\alpha,\beta}(s - d, s)$

For $d \in \mathbb{N}_0$ and $n := 2a + 1 \leq d$ where $a \in \mathbb{Z}$ the condition (3.9) is fulfilled when choosing:

$$p = q := 2d - n + 1 \in \mathbb{N}.$$

By inserting (3.2) into (3.1) and with the remark above Lemma 3.1 one obtains:

$$\begin{aligned} &\lim_{s\to n/2} (s - n/2) H_{\alpha,\beta}(s - d, s) \\ &\quad = \lim_{s\to n/2} \frac{\Gamma(2s - d)(s - n/2)}{\Gamma(s - d)\Gamma(s)} \int_0^{\beta-\alpha} (\beta - \alpha - x)^{s-d-1} x^{s-1} \frac{P^{p,p}_{\alpha,\beta}(s, x)}{(\beta - \alpha)^{2s-d-1}} \mathrm{d}x. \end{aligned} \tag{3.10}$$

Since the Gamma function has simple poles at all non-positive integers $-n$ where $n \in \mathbb{N}_0$ with residue $(-1)^n/n!$ we can write:

$$\Gamma(2s - d) = \frac{1}{2s - n} \cdot \frac{(-1)^{d-n}}{(d - n)!} + h(s)$$

where $h(s)$ is analytic in a neighborhood of $n/2$. Now, (3.10) implies after a straightforward calculation:

$$\begin{aligned} \mathrm{Res}\left(H_{\alpha,\beta}(s - d, s), s = \frac{n}{2}\right) &= \frac{(-1)^d}{(d - n)!} \sum_{j=0}^{2d-n} \frac{(\beta - \alpha)^{2d-n+j+1}}{(j + d)!} \\ &\quad \times \left[(-1)^{j+n} b^{\alpha,\beta,p}_j\left(\frac{n}{2}\right) \kappa\,(d + 1, j + n - 2d) - a^{\alpha,\beta,p}_j\left(\frac{n}{2}\right) \kappa\,(1, j + n - d)\right] \end{aligned} \tag{3.11}$$

where for integers a and b with $a \geq n - d$ and $b \geq n - 2d$ we define:

$$\kappa\,(a, b) := \frac{\Gamma\left(d - \frac{n}{2} + a\right)\Gamma\left(d - \frac{n}{2} + b\right)}{2\Gamma\left(\frac{n}{2} - d\right)\Gamma\left(\frac{n}{2}\right)} = \frac{1}{2} \prod_{t=0}^{a-n+d-1} \left(\frac{n}{2} + t\right) \prod_{t=-d}^{b-n+d-1} \left(\frac{n}{2} + t\right).$$

From (3.5) and (A.2) with the Bernoulli polynomials B_k, $k \in \mathbb{N}_0$ and $B_k \equiv 0$ for $k < 0$ one obtains for $j = 0, \dots 2d - n$:

$$a^{\alpha,\beta,p}_j\left(\frac{n}{2}\right) = \frac{(d - n)!(-1)^{j+1}}{(\alpha - \beta)^{2d-n+1+j}} \sum_{l=0}^{j} \binom{2d - n + j - l}{j - l} \frac{B_{d-n+1-l}(\alpha)}{(d - n + 1 - l)!l!} (\beta - \alpha)^l.$$

A similar formula holds for the coefficients $b^{\alpha,\beta,p}_j(n/2)$ and after inserting these expression into (3.11) we derive:

Proposition 3.1. *For $d \in \mathbb{N}_0$ and integers $n = 2a+1 \leq d$ where $a \in \mathbb{Z}$ the meromorphic the function $H_{\alpha,\beta}(s-d,s)$ has simple poles at $s = n/2$ and $s = (d+1)/2$. Moreover,*

$$R_{\alpha,\beta,d}\left(\frac{n}{2}\right) = (-1)^{n+d+1}\sum_{j=0}^{2d-n}\sum_{l=0}^{j}\binom{2d-n+j-l}{j-l}\frac{(\beta-\alpha)^l}{(j+d)!l!}$$
$$\times\left[(-1)^l\frac{B_{d-n+1-l}(\beta)}{(d-n+1-l)!}\kappa\,(d+1, j+n-2d) + \frac{B_{d-n+1-l}(\alpha)}{(d-n+1-l)!}\kappa\,(1, j+n-d)\right]$$

is the residue at $s = n/2$. In $s = (d+1)/2$ the residue of $H_{\alpha,\beta}(s-d,s)$ equals $1/2$.

Proof. The proof of the last assertion remains: For $\mathrm{Re}(y) > 0$ let $h(\cdot, y)$ be an entire function such that:

$$\zeta\,(2s-d, y) = \frac{1}{2s-d-1} + h(s,y). \tag{3.12}$$

By applying a decomposition similar to the one in (3.2) to $h(s, x+\alpha)$ instead of the zeta function there it can be checked that:

$$\frac{\Gamma(2s-d)}{\Gamma(s-d)\Gamma(s)}\int_0^{\beta-\alpha}(\beta-\alpha-x)^{s-d-1}x^{s-1}h\,(s, x+\alpha)\,\mathrm{d}x$$

is holomorphic in a neighborhood of $s = (d+1)/2$. By using (3.12) in (3.1) it follows:

$$\lim_{s\to\frac{d+1}{2}}\left(s - \frac{d+1}{2}\right)H_{\alpha,\beta}(s-d,s) = \frac{1}{2}. \quad \square$$

Example 3.1. We explicitly write the second residues of $H_{\alpha,\beta}(s-d,s)$ for $d = 0, 1, 2$:

(i) $R_{\alpha,\beta,0}(-1/2) = -\frac{(\beta-\alpha)^2}{16}$,
(ii) $R_{\alpha,\beta,1}(1/2) = \frac{\alpha-\beta}{4}$,
(iii) $R_{\alpha,\beta,2}(1/2) = \frac{3}{16}(\beta-\alpha)^2$.

In particular, these expressions only depend on the difference between α and β and vanish for $\alpha \to \beta$.

3.2. The values $H_{\alpha,\beta}(-d, 0)$

From now on we consider the point $s = 0$ in which $H_{\alpha,\beta}(s-d,s)$ is holomorphic. By choosing $p := d+1$ and $q := 1$ in (3.2) condition (3.9) holds for $n := 0$. Therefore it follows from the remark above Lemma 3.1 that:

$$H_{\alpha,\beta}\,(-d,0) = \lim_{s\downarrow 0}\frac{1}{(\beta-\alpha)^{2s-d-1}}\frac{\Gamma(2s-d)}{\Gamma(s-d)\Gamma(s)}\int_0^{\beta-\alpha}(\beta-\alpha-x)^{s-d-1}x^{s-1}P^{d+1,1}_{\alpha,\beta}(s,x)\mathrm{d}x. \tag{3.13}$$

According to the decomposition (3.3) we write:

$$H_{\alpha,\beta}\,(-d,0) = \lim_{s\downarrow 0}\left\{J(s) + \sum_{m=0}^{d}I_m(s)\right\}. \tag{3.14}$$

Using Lemma A.1 (4), the meromorphic function $J(s)$ is given by:

$$\begin{aligned} J(s) :&= \frac{(-1)^{d+1}}{(\beta-\alpha)^{2s-d-1}}A^{d+1,1}_{\alpha,\beta}(s)\frac{\Gamma(2s-d)}{\Gamma(s-d)\Gamma(s)}(\beta-\alpha)^{2s}\frac{\Gamma(s+1)\Gamma(s)}{\Gamma(2s+1)} \\ &= \frac{1}{2}\zeta\,(2s-d,\alpha)\prod_{j=1}^{d}\frac{s-j}{2s-j}. \end{aligned} \tag{3.15}$$

For $m \in \{0, \ldots, d\}$ it follows from Lemma A.1, (4):

$$I_m(s) = b_m^{\alpha,\beta,1}(s)(-1)^m\,(\beta-\alpha)^{m+1}\,\frac{s}{2s-d+m}\prod_{j=0}^{m-1}\frac{s-d+j}{2s-d+j}.$$

After inserting the expression (3.7) for $b_m^{\alpha,\beta,1}(s)$ one has:

$$I_m(s) = \sum_{l=0}^{m}\left\{\frac{(\beta-\alpha)^{m-l}}{(m-l)!}\,\frac{s}{2s-d+m}\zeta\,(2s-d+m-l,\beta)\prod_{j=0}^{m-l-1}(2s-d+j)\prod_{j=0}^{m-1}\frac{s-d+j}{2s-d+j}\right\} \tag{3.16}$$

where $m \in \{0, \ldots, d\}$ (“empty products” $= 1$).

Proposition 3.2. *For $\beta > \alpha > 0$:*

$$H_{\alpha,\beta}\,(-d,0) = \zeta\,(-d,\alpha) + \frac{1}{2}\frac{(\alpha-\beta)^{d+1}}{d+1}.$$

Proof. Note that $J(0) = \frac{1}{2}\zeta(-d,\alpha)$ and $I_m(0) = 0$ for $m \in \{0, \ldots, d-1\}$. Moreover,

$$I_d(0) = \frac{1}{2}\sum_{l=0}^{d}\binom{d}{l}(\alpha-\beta)^{d-l}\,\zeta\,(-l,\beta) = \frac{1}{2}\zeta\,(-d,\alpha) + \frac{1}{2}\frac{(\alpha-\beta)^{d+1}}{d+1}.$$

Here we used Lemma A.1, (5) in the second equation. □

Example 3.2. Applying Lemma A.1, (1) and the explicit expressions for the Bernoulli polynomials given in the Appendix one has:

(i) $H_{\alpha,\beta}\,(0,0) = \frac{1}{2}(1-\alpha-\beta)$,
(ii) $H_{\alpha,\beta}\,(-1,0) = -\frac{1}{2}\left(\alpha^2-\alpha+\frac{1}{6}\right)+\frac{1}{4}\,(\alpha-\beta)^2$,
(iii) $H_{\alpha,\beta}\,(-2,0) = -\frac{1}{3}\alpha\,(\alpha-1)\left(\alpha-\frac{1}{2}\right)+\frac{1}{6}\,(\alpha-\beta)^3$.

3.3. The values $H'_{\alpha,\beta}(-d,0)$

From Lemma 3.1 it is clear, that the expression

$$\frac{\Gamma(2s-d)}{\Gamma(s-d)\Gamma(s)}\int_0^{\beta-\alpha}(\beta-\alpha-x)^{s-d-1}\,x^{s-1}F_{\alpha,\beta}^{d+1,1}(s,x)\mathrm{d}x$$

vanishes to second order at $s = 0$. Using the notations of (3.14) we find:

$$H'_{\alpha,\beta}\,(-d,0) = \frac{\partial}{\partial s}\left\{J(s)+\sum_{m=0}^{d}I_m(s)\right\}_{|s=0}. \tag{3.17}$$

In the following we write $\zeta'(s,x) := \frac{\partial}{\partial s}\zeta(s,x)$. By a direct calculation it follows from (3.15) and (3.16), that:

$$\frac{\partial J}{\partial s}(0) = \frac{1}{2}\zeta\,(-d,\alpha)\sum_{j=1}^{d}\frac{1}{j}+\zeta'\,(-d,\alpha)\,. \tag{3.18}$$

For $m \in \{0, \ldots, d-1\}$:

$$\frac{\partial I_m}{\partial s}(0) = \sum_{l=0}^{m}\frac{(\alpha-\beta)^{m-l}}{-d+m}\binom{d}{m-l}\zeta\,(-d+m-l,\beta)\,. \tag{3.19}$$

For $m = d$ one has:

$$I_d(s) = \frac{1}{2}\sum_{l=0}^{d}\frac{(\beta-\alpha)^{d-l}}{(d-l)!}\zeta\,(2s-l,\beta)\underbrace{\prod_{j=0}^{d-l-1}(s-d+j)\prod_{j=d-l}^{d-1}\frac{s-d+j}{2s-d+j}}_{=:h(s)}$$

where

$$\frac{\partial h}{\partial s}(0) = (-1)^{d-l}\frac{d!}{l!}\left\{\sum_{j=1}^{l}\frac{1}{j}-\sum_{j=l+1}^{d}\frac{1}{j}\right\}.$$

Hence, if the empty sum is interpreted as 0:

$$\frac{\partial I_d}{\partial s}(0) = \sum_{l=0}^{d}(\alpha-\beta)^{d-l}\binom{d}{l}\left\{\zeta'\,(-l,\beta)+\frac{1}{2}\left[\sum_{j=1}^{l}\frac{1}{j}-\sum_{j=l+1}^{d}\frac{1}{j}\right]\zeta\,(-l,\beta)\right\}.$$

According to (3.17):

$$\begin{aligned}H'_{\alpha,\beta}\,(-d,0) \;=\; &\frac{1}{2}\zeta\,(-d,\alpha)\sum_{j=1}^{d}\frac{1}{j}+\zeta'\,(-d,\alpha)+\sum_{m=0}^{d-1}\sum_{l=0}^{m}\frac{(\alpha-\beta)^{m-l}}{-d+m}\binom{d}{m-l}\zeta\,(-d+m-l,\beta)\\ &+\sum_{l=0}^{d}(\alpha-\beta)^{d-l}\binom{d}{l}\left\{\zeta'\,(-l,\beta)+\frac{1}{2}\left[\sum_{j=1}^{l}\frac{1}{j}-\sum_{j=l+1}^{d}\frac{1}{j}\right]\zeta\,(-l,\beta)\right\}.\end{aligned}$$

For further simplification of this expression note that:

$$\begin{aligned}\sum_{m=0}^{d-1}\sum_{l=0}^{m}\frac{(\alpha-\beta)^{m-l}}{-d+m}\binom{d}{m-l}\zeta\,(-d+m-l,\beta) &= \sum_{m=0}^{d-1}\sum_{l=d-m}^{d}\binom{d}{l}\frac{(\alpha-\beta)^{d-l}}{-d+m}\zeta\,(-l,\beta)\\ &= \sum_{l=1}^{d}\binom{d}{l}(\alpha-\beta)^{d-l}\,\zeta\,(-l,\beta)\sum_{m=d-l}^{d-1}\frac{1}{-d+m}\\ &= -\sum_{l=0}^{d}\binom{d}{l}(\alpha-\beta)^{d-l}\,\zeta\,(-l,\beta)\sum_{j=1}^{l}\frac{1}{j}.\end{aligned}$$

Thus we arrive at:

$$\begin{aligned}H'_{\alpha,\beta}\,(-d,0) \;=\; &\frac{1}{2}\left\{\zeta\,(-d,\alpha)-\sum_{l=0}^{d}\binom{d}{l}(\alpha-\beta)^{d-l}\,\zeta\,(-l,\beta)\right\}\sum_{j=1}^{d}\frac{1}{j}\\ &+\zeta'\,(-d,\alpha)+\sum_{l=0}^{d}\binom{d}{l}(\alpha-\beta)^{d-l}\,\zeta'\,(-l,\beta)\,.\end{aligned}$$

Finally, by using Lemma A.1, (5) it has been shown:

Proposition 3.3. *For $\beta > \alpha > 0$ and $d \in \mathbb{N}_0$:*

$$H'_{\alpha,\beta}\,(-d,0) = -\frac{(\alpha-\beta)^{d+1}}{2d+2}\sum_{j=1}^{d}\frac{1}{j}+\zeta'\,(-d,\alpha)+\sum_{l=0}^{d}\binom{d}{l}(\alpha-\beta)^{d-l}\,\zeta'\,(-l,\beta)\,.$$

Example 3.3. We write the formulas in Proposition 3.3 explicitly for $d = 0, 1, 2$:

(i) $H'_{\alpha,\beta}(0,0) = \zeta'(0,\alpha)+\zeta'(0,\beta)$ (*generalized Lerch formula*),

(ii) $H'_{\alpha,\beta}(-1,0) = -\frac{(\alpha-\beta)^2}{4} + \zeta'(-1,\alpha) + (\alpha-\beta)\zeta'(0,\beta) + \zeta'(-1,\beta)$,

(iii) $H'_{\alpha,\beta}(-2,0) = -\frac{(\alpha-\beta)^3}{4} + \zeta'(-2,\alpha) + (\alpha-\beta)^2\zeta'(0,\beta) + 2(\alpha-\beta)\zeta'(-1,\beta) + \zeta'(-2,\beta)$.

4. Examples and applications

For a variety of fix point free groups $G \subset SO(2N+2)$ we derive explicit expressions for the zeta regularized determinant $\det \Delta_{M_G}$ of the spherical space form $\mathbb{S}^{2N+1}/G$. Apart from the 3-dimensional lens spaces which have been treated in [13] we also consider cases of higher dimensions and the *binary dihedral groups* $\mathbf{D}_m^*$ for $m \in \mathbb{N}$ and $N = 1$ which are not lens spaces (cf. the classification in [14,19]).

4.1. 3-dimensional spherical space forms

For $N = 1$ (3-dimensional case) we first calculate all possible residues $C_k(g,\gamma)$ in (2.9) for $g = I$ and fix point free elements $g \in SO(4)$. As an example we derive the well-known formulas for the zeta-regularized determinants for the sphere $\mathbb{S}^3$ and the real projective space $\mathbb{R}P^3$. Four cases can be distinguished:

Case I: For $g = I$ one has $\gamma = 1$ and according to (2.1) or (2.9): $C_k(I,1) = \mathbf{m}_k = (k+1)^2$. With our notations in (2.4) this shows that:

$$\zeta_{\mathbb{S}^3}(s) = J_{I,1}(s) = H_{1,3}(s-2,s) + 2H_{1,3}(s-1,s) + H_{1,3}(s,s).$$

Example 4.1 (*cf. [15]*). According to Proposition 3.3 (Example 3.3):

$$\zeta'_{\mathbb{S}^3}(0) = \zeta'(-2) + \zeta'(-2,3) + 2\zeta'(-1) - 2\zeta'(-1,3) + \log\frac{1}{\pi}$$

where we used $\zeta'(0) = -\log\sqrt{2\pi}$. Moreover, from

$$\zeta'(-m,1+n) = \zeta'(-m) + \sum_{r=2}^{n} r^m \log r \tag{4.1}$$

together with[2] $\zeta'(-2) = -\frac{\zeta(3)}{4\pi^2}$ we find for the zeta-regularized determinant of $\mathbb{S}^3$ the well-known expression:

$$\log\det\Delta_{\mathbb{S}^3} = \frac{\zeta(3)}{2\pi^2} + \log\pi.$$

Case II: For $g = -I$ one has $C_k(-I,-1) = (-1)^k(k+1)^2$. According to (2.4):

$$J_{-I,-1}(s) = J_1(s) - J_{\frac{1}{2}}(s)$$

where we define for $\alpha > 0$:

$$J_\alpha(s) = \frac{1}{4^{s-1}}H_{\alpha,1+\alpha}(s-2,s) + \frac{1}{4^{s-1}}H_{\alpha,1+\alpha}(s-1,s) + \frac{1}{4^s}H_{\alpha,1+\alpha}(s,s).$$

Therefore, the spectral zeta function of $\mathbb{R}P^3$ can be written as:

$$\zeta_{\mathbb{R}P^3}(s) = \frac{\zeta_{\mathbb{S}^3}(s)}{2} + \frac{1}{2}\left\{J_1(s) - J_{\frac{1}{2}}(s)\right\}. \tag{4.2}$$

and $J_{-I,-1} = J_1 - J_{\frac{1}{2}}$ has an extension to an entire function according to Proposition 2.1.

[2] Cf. [13] p. 1502 or differentiate $\zeta(s) = 2^s\pi^{s-1}\Gamma(1-s)\zeta(1-s)\sin(\frac{s\pi}{2})$ in [18] p. 275 at $s = -2$.

Example 4.2 (*cf.* [13]). By a direct calculation using $\zeta'(s, 1+\alpha) = \zeta'(s,\alpha)+\alpha^{-s}\log\alpha$ together with Propositions 3.2 and 3.3:

$$\frac{\partial J_\alpha}{\partial s}(0) = 8\zeta'(-2,\alpha) + 2\zeta'(0,\alpha) + (1-2\alpha)^2\log\alpha + \frac{4}{3}\alpha\left(\alpha-\frac{1}{2}\right)\left(\alpha+\frac{1}{2}\right)\log 4.$$

Hence, with $\zeta'(0) - \zeta'(0,\frac{1}{2}) = \log\frac{1}{\sqrt{\pi}}$:

$$\frac{\partial}{\partial s}J_{-I,-1}(0) = 8\left[\zeta'(-2) - \zeta'\left(-2,\frac{1}{2}\right)\right] + \log\frac{4}{\pi} = \frac{4}{\pi^2}\int_0^{\frac{\pi}{2}} z(z-\pi)\cot(z)\mathrm{d}z + \log\frac{8}{\pi}$$

where in the second equation we have used the following identity in [13], p. 1503:

$$\zeta'\left(-2,\frac{1}{2}\right) = \zeta'(-2) - \frac{\log 2}{8} - \frac{1}{2\pi^2}\int_0^{\frac{\pi}{2}} z(z-\pi)\cot(z)\mathrm{d}z.$$

Finally, by using (4.2) it follows:

$$\log\det\Delta_{\mathbb{R}P^3} = \frac{\log\det\Delta_{\mathbb{S}^3}}{2} - \frac{2}{\pi^2}\int_0^{\frac{\pi}{2}} z(z-\pi)\cot(z)\mathrm{d}z - \frac{1}{2}\log\frac{8}{\pi}.$$

Case III: Let $g \in SO(4)$ be of order $q_g > 2$ with eigenvalues $\{\gamma,\gamma,\overline{\gamma},\overline{\gamma}\}$. By a straightforward calculation and for $\delta \in \{\gamma,\overline{\gamma}\}$:

$$C_k(g,\delta) = (k+1)\frac{\gamma^{-k}}{1-\gamma^2}. \tag{4.3}$$

With our notations in Proposition 2.2 and for $\nu = 0,\ldots,q_g-1$:

$$\alpha_0(g,\nu) = 2\mathrm{Re}\left(\frac{\gamma^{-\nu}}{1-\gamma^2}\right) \qquad \text{and} \qquad \alpha_1(g,\nu,) = 2q_g\mathrm{Re}\left(\frac{\gamma^{-\nu}}{1-\gamma^2}\right).$$

Case IV: Let $g \in SO(4)$ be of order $q_g > 2$ with eigenvalues $\{\gamma,\overline{\gamma},\mu,\overline{\mu}\} \in \mathbb{S}^1 \setminus \{-1,1\}$ such that $\gamma \notin \{\mu,\overline{\mu}\}$. Then it holds by a straightforward calculation, cf. (4.8):

$$C_k(g,\gamma) = \frac{\gamma^{-k}}{(\gamma-\mu)(\gamma-\overline{\mu}).}$$

Hence, with the notations of Proposition 2.2 and $C_k(g,\overline{\gamma}) = \overline{C_k(g,\gamma)}$ one has:

$$\alpha_0(g,\nu) = 2\mathrm{Re}\left\{\frac{\gamma^{-(\nu+1)}-\mu^{-(\nu+1)}}{\gamma-\mu}\frac{1}{1-\overline{\gamma\mu}}\right\} \qquad \text{and} \qquad \alpha_1(g,\nu) = 0$$

for $k = q_g\ell+\nu$ and $\nu = 0,\ldots,q_g-1$.

4.2. Lens spaces

Let $q, N \in \mathbb{N}$ and set $\gamma := \exp(2\pi\sqrt{-1}/q)$. With integers $p_0, p_1, \ldots, p_N$ prime to q and the identification $\mathbb{C}^{N+1} \cong \mathbb{R}^{2N+2}$ we define an isometry g of $\mathbb{R}^{2N+2}$ by:

$$g : (z_0, z_1, \ldots, z_N) \to \left(\gamma^{p_0}z_0, \gamma^{p_1}z_1, \ldots, \gamma^{p_N}z_N\right). \tag{4.4}$$

Then g generates a cyclic group $G = \{g^k\}_{k=0,\ldots,q-1}$ in $SO(2N+2)$ and all $I \neq g \in G$ act fix point freely on $\mathbb{S}^{2N+1}$. The spherical space form

$$L(q : p_0,\ldots,p_N) := M_G = \mathbb{S}^{2N+1}/G$$

is called a *lens space*. By Theorem 2.1, the generating function converges on $|z| < 1$ to

$$F_G(z) = \frac{1}{q} \sum_{l=0}^{q-1} \frac{1 - z^2}{\prod_{i=0}^{N} (1 - \gamma^{p_i l} z)(1 - \gamma^{-p_i l} z)}$$

which has (for $l = 0$) a pole of order $2N + 1$ at $z = 1$ with coefficient $2/q$ in the Laurent expansion.

4.2.1. The case $p_0 = p_1 = \cdots = p_N = 1$

In particular, by choosing $p_0 = p_1 = \cdots = p_N = 1$ one finds:

$$\frac{z^2 - 1}{z^{k+1} \det(I - g^j z)} = \frac{1}{(z - \gamma^{-j})^{N+1} (z - \gamma^j)^{N+1}} \left\{ \frac{1}{z^{k-1}} - \frac{1}{z^{k+1}} \right\}.$$

Applying (2.7) with $\rho = 0$ and $\rho = \gamma^{-j}$, and $\tilde{\gamma} = \gamma^j$ $(j \neq 0)$:

$$\frac{1}{z^{k-1} (z - \gamma^{-j})^{N+1} (z - \gamma^j)^{N+1}} = \sum_{n_1, n_2 = 0}^{\infty} (-1)^{n_1 + n_2} \binom{k - 2 + n_1}{n_1} \binom{N + n_2}{n_2}$$
$$\times \gamma^{-j(k-1+n_1)} \left(\gamma^j - \gamma^{-j} \right)^{-(N+1+n_2)} \left(z - \gamma^j \right)^{n_1 + n_2 - N - 1}.$$

By using these expressions and as a generalization of *Case* III in Section 4.1 one readily verifies for integers $N \geq 1$:

Lemma 4.1. *For all $k \in \mathbb{N}_0$ and $\delta := \gamma^j$:*

$$C_k\left(g^j, \delta\right) = \sum_{n=0}^{N-2} \binom{N + n}{n} \left\{ \binom{k + N - 2 - n}{N - n} - \delta^{-2} \binom{k + N - n}{N - n} \right\}$$
$$\times \frac{(-1)^{n+1} \delta^{2(n+1) - k}}{(1 - \delta^2)^{n+N+1}} - (-1)^N (k + 1) \binom{2N - 1}{N - 1} \frac{\delta^{2(N-1) - k}}{(1 - \delta^2)^{2N-1}}.$$

In particular, $\overline{C_k(g^j, \delta)} = C_k(g^j, \delta^{-1})$.

Assume that q is an odd integer. Let g and $G = \{g^k\}_{k=0,\ldots,q-1}$ be as above. Then it follows from Proposition 2.1:

$$\mathbf{m}_k(M_G) = \frac{\mathbf{m}_k}{q} + \frac{2}{q} \mathrm{Re} \sum_{j=1}^{q-1} C_k\left(g^j, \gamma^j\right). \tag{4.5}$$

We explicitly treat the case $N = 1$ (=3-dimensional lens space). We find from (4.3) or Lemma 4.1 that:

$$C_k\left(g^j, \delta\right) = (k + 1) \frac{\delta^{-k}}{1 - \delta^2}.$$

Therefore, applying Lemma A.3 one has for the dimensions of the eigenspaces:

$$\mathbf{m}_k(M_G) = \frac{\mathbf{m}_k}{q} + (k + 1) r_k$$

where for $k \in \mathbb{N}$ the numbers

$$r_k := \frac{1}{q} \left[q + 1 - 2\beta_q^{(1)}(-k) \right] \in \frac{1}{q} \mathbb{Z}$$

are q-periodic in k. Here the expressions $\beta_q^{(1)}(-k)$ are defined in (A.3) of the Appendix. By writing $k = \ell q + \nu$ where $0 \leq \nu \leq q - 1$ and $\ell \in \mathbb{N}_0$ we obtain from a decomposition similar to the one in (2.11):

$$\zeta_{M_G}(s) = \frac{\zeta_{\mathbb{S}^3}(s)}{q} + \frac{1}{q^{2s}} \sum_{\nu=1}^{q} r_\nu \left[q H_{\frac{\nu}{q}, \frac{\nu+2}{q}}(s - 1, s) + H_{\frac{\nu}{q}, \frac{\nu+2}{q}}(s, s) \right].$$

According to Proposition 2.1 the second term on the right hand side extends to an entire function. By evaluating its residue in $s = 1$ and using Proposition 3.1 one has the identity:

$$\sum_{\nu=1}^{q} r_\nu = 0. \tag{4.6}$$

Theorem 4.1 (*[13]*). *The zeta-regularized determinant of M_G is given by:*

$$\log\det \Delta_{M_G} = \frac{\log\det \Delta_{\mathbb{S}^3}}{q} - \sum_{\nu=1}^{q} r_\nu A_q(\nu) \tag{4.7}$$

where in terms of the Hurwitz Zeta function and its derivatives:

$$\begin{aligned} A_q(\nu) = {} & q\left[\zeta'\left(-1, \frac{\nu}{q}\right) + \zeta'\left(-1, \frac{\nu+2}{q}\right)\right] - \frac{1}{q} + \zeta'\left(0, \frac{\nu}{q}\right) - \zeta'\left(0, \frac{\nu+2}{q}\right) \\ & - 2\left[q\zeta\left(-1, \frac{\nu}{q}\right) + \zeta\left(0, \frac{\nu}{q}\right)\right] \log q. \end{aligned}$$

Proof. By Propositions 3.2 and 3.3:

$$A_q(\nu) = \frac{\partial}{\partial s}\left[\frac{1}{q^{2s-1}} H_{\frac{\nu}{q}, \frac{\nu+2}{q}}(s-1, s) + \frac{1}{q^{2s}} H_{\frac{\nu}{q}, \frac{\nu+2}{q}}(s, s)\right]_{|s=0}. \quad \square$$

Remark 4.1. From (4.6) together with Proposition B.1 it directly follows that the sum on the right hand side of (4.7) can completely be written in terms of the following multiple Gamma functions: $\Gamma_0(x) = x^{-1}$, $\Gamma_1 := \Gamma$ and $\Gamma_2 := G$ (Barnes's G-function).

Finally, we want to derive a second and more simple expression for $\mathbf{m}_k(M_G)$. Using (A.7) of Lemma A.4 shows that:

$$\mathbf{m}_k(M_G) = (k+1)\gamma_q(k) \le \frac{(k+1)(k+q+2)}{q}.$$

Here the integers $\gamma_q(k)$ are defined in (A.6) of the Appendix.

Example 4.3. In the case $N = 2$ we only calculate the dimensions of the eigenspaces of the Laplacian. However, the derivative of the zeta-regularized determinant can be derived in a way similar to our calculation above. According to Lemma 4.1 and for an eigenvalue δ of $g^j \neq I$:

$$C_k(g^j, \delta) = -\binom{k+3}{2}\frac{\delta^{2-k}}{(1-\delta^2)^3} + \binom{k+2}{2}\frac{\delta^{-k}}{(1-\delta^2)^3}.$$

This together with (4.5) and Lemma A.3 shows that:

$$\mathbf{m}_k(M_G) = \frac{\mathbf{m}_k}{q} + \frac{k+2}{q^3}\left[(k+3)\beta_q^{(3)}(2-k) - (k+1)\beta_q^{(3)}(-k) - \frac{q^2}{4}(q+1)^3\right].$$

4.2.2. The case: q odd prime and $1 = p_0 < \cdots < p_N < q$

Let $q \in \mathbb{N}$ be an odd prime number and assume that $q > N + 1$. With integers

$$1 = p_0 < \cdots < p_N < q$$

we consider the lens space $M_G := L(q : p_0, \ldots, p_N)$. For γ and g as in (4.4) and with (2.6) it follows that the assignment:

$$\mathbb{C} \ni z \mapsto \frac{z^2 - 1}{z^{k+1}\det(I - g^l z)}$$

has simple poles at γ^{lp_j} for $j = 0, \dots N$ and $l = 1, \dots q-1$. One readily verifies for the residues defined in Proposition 2.1:

$$C_k\left(g^l, \gamma^{lp_j}\right) = \frac{1}{\gamma^{lkp_j}} \prod_{\substack{m=0\\ m\neq j}}^{N} \frac{1}{\left[1-\gamma^{l(p_j-p_m)}\right]\left[1-\gamma^{l(p_j+p_m)}\right]}. \tag{4.8}$$

In particular, we note from this expression that:

$$C_k\left(g^l, \gamma^{lp_j}\right) = \overline{C_k\left(g^l, \gamma^{-lp_j}\right)}.$$

In order to calculate the dimensions $\mathbf{m}_k(M_G)$ of the eigenspaces of the Laplacian Δ_{M_G} more explicitly we need the following result:

Lemma 4.2. *Let $\{\gamma_0, \dots, \gamma_m\} \subset \mathbb{S}^1$ be q-th roots of unity and $\gamma_j \neq 1$. Then for $k \in \mathbb{Z}$:*

$$\sum_{r=1}^{q-1} \gamma_0^{rk} \prod_{l=1}^{m} \frac{1}{1-\gamma_l^r} = \frac{(-1)^m}{q^m} \sum_{i_1,\dots i_m=1}^{q} i_1 i_2 \cdots i_m \times \begin{cases} q-1 & \text{if } \gamma_1^{i_1-1}\gamma_2^{i_2-1}\cdots\gamma_m^{i_m-1} = \gamma_0^{-k} \\ -1 & \text{else.} \end{cases}$$

Proof. According to Lemma A.2 one can write:

$$\prod_{l=1}^{m} \frac{1}{1-\gamma_l^r} = \frac{(-1)^m}{q^m} \sum_{i_1,\dots i_m=1}^{q} i_1 i_2 \cdots i_m \gamma_1^{r(i_1-1)} \cdots \gamma_s^{r(i_m-1)}.$$

After multiplying by γ_0^{rk} and summing over r the assertion follows. □

By writing $\widehat{i_j}$ we mean as usual that i_j does not appear in the summation below. We define the integers:

$$\delta_{q,k}(p_j) := \sum_{\substack{i_0,\dots \widehat{i_j},\dots i_N=1\\ s_0,\dots \widehat{s_j}\cdots s_N=1}}^{q} i_0 s_0 \cdots \widehat{i_j}\widehat{s_j} \cdots i_N s_N \times \begin{cases} 1 & \text{if } p_j\left(\sum_{l\neq j}[i_l+s_l]-k-2N\right) \\ & \quad \equiv \prod_{l\neq j} p_l(i_l-s_l) \bmod q \\ 0 & \text{else.} \end{cases} \tag{4.9}$$

Then (4.9) is q-periodic in the variable k and from (4.8) together with Lemma 4.2 it can be deduced that:

$$\sum_{r=1}^{q-1} C_k\left(g^r, \gamma^{rp_j}\right) = \frac{\delta_{q,k}(p_j)}{q^{2N-1}} - \frac{(q+1)^{2N}}{4^N}.$$

For $G := \{g^k\}_{k=0,\dots,q-1}$ and according to Proposition 2.1 one obtains for the dimension of the eigenspaces:

$$\mathbf{m}_k(M_G) = \frac{\mathbf{m}_k}{q} + \frac{2}{q}\sum_{r=1}^{q-1}\sum_{j=0}^{N} C_k\left(g^r, \gamma^{rp_j}\right) = \frac{1}{q}\left[\mathbf{m}_k - \frac{N+1}{2^{2N-1}}(q+1)^{2N}\right] + \frac{2}{q^{2N}}\sum_{j=0}^{N}\delta_{q,k}(p_j). \tag{4.10}$$

In particular, if we write $k = \ell q + \nu$ it follows that $\mathbf{m}_k(M_G)$ has the form

$$\mathbf{m}_k(M_G) = \frac{\mathbf{m}_{\ell q+\nu}}{q} + t_\nu,$$

where the numbers t_ν are defined by:

$$t_\nu := \frac{2}{q^{2N}}\left[\sum_{j=0}^{N}\delta_{q,\nu}(p_j) - q^{2N-1}(N+1)\frac{(q+1)^{2N}}{4^n}\right] \in \frac{2}{q^{2N}}\mathbb{Z}.$$

By a direct calculation using $t_q = t_0$ it follows that:

$$\zeta_{M_G}(s) = \frac{\zeta_{\mathbb{S}^{2N+1}}(s)}{q} + \frac{1}{q^{2s}} \sum_{\nu=1}^{q} t_\nu H_{\frac{\nu}{q}, \frac{\nu+2N}{q}}(s, s) \tag{4.11}$$

and the sum on the right hand side defines an entire function. By forming the derivative:

$$\zeta'_{M_G}(s) = \frac{\zeta'_{\mathbb{S}^{2N+1}}(s)}{q} + \sum_{\nu=1}^{q} t_\nu \left[H'_{\frac{\nu}{q}, \frac{\nu+2N}{q}}(s, s) - 2H_{\frac{\nu}{q}, \frac{\nu+2N}{q}}(s, s) \log q \right].$$

Hence, by Propositions 3.2 and 3.3 we obtain for $\det \Delta_{M_G}$:

Theorem 4.2. *Let q be an odd prime number and $1 = p_0 < \cdots < p_N < q$, then:*

$$\log\det \Delta_{M_G} = \frac{\log\det \Delta_{\mathbb{S}^{2N+1}}}{q} - \sum_{\nu=1}^{q} t_\nu B_q(\nu)$$

where

$$B_q(\nu) = \zeta'\left(0, \frac{\nu}{q}\right) + \zeta'\left(0, \frac{\nu+2N}{q}\right) - 2\zeta\left(0, \frac{\nu}{q}\right) \log q + \frac{2N \log q}{q}.$$

Remark 4.2. By applying Lemma A.1, (1) and (2) the numbers $B_q(\nu)$ are real and can be expressed in terms of the Gamma function as:

$$B_q(\nu) = \log \frac{\Gamma\left(\frac{\nu}{q}\right)\Gamma\left(\frac{\nu+2N}{q}\right)}{2\pi} - \frac{q - 2\nu - 2N}{q} \log q.$$

Moreover, we remark that by calculating the residues on both sides of (4.11) at $s = 1/2$ and by Proposition 3.1 it follows that

$$\sum_{\nu=1}^{q} t_\nu = 0.$$

4.2.3. *Spherical space forms via the dihedral group*

Consider the *dihedral group* $\mathbf{D}_m$ of degree m (and order $2m$) in $SO(3)$ (cf. [19]) which is generated by the matrices:

$$A := \begin{pmatrix} \cos\left(\frac{2\pi}{m}\right) & -\sin\left(\frac{2\pi}{m}\right) & 0 \\ \sin\left(\frac{2\pi}{m}\right) & \cos\left(\frac{2\pi}{m}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then A and B satisfy the relations $A^m = B^2 = 1$ and $BAB^{-1} = A^{-1}$. Let

$$\mathbb{H} = \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} : a, b, c, d \in \mathbb{R}\}$$

denote the field of quaternions with basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ over $\mathbb{R}$ and with the usual relations: $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$, $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$ and $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$. The subset

$$\mathbb{H}' := \left\{ q \in \mathbb{H} : |q|^2 = q\bar{q} = 1 \right\}$$

of $\mathbb{H}$ forms the multiplicative group of *unit quaternions* which is isomorphic to $SU(2)$. The real 3-dimensional subspace of $\mathbb{H}$ spanned by $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is denoted by $\mathbb{H}_0$. There is a map π which is onto and defined by:

$$\pi : \mathbb{H}' \to SO(3) : \pi(q)p := qp\bar{q}$$

where $q \in \mathbb{H}'$ and $p \in \mathbb{H}_0$. Solving $\pi\left(q_{A,j}\right) = A^j$ for $j = 0, \dots m-1$ and $\pi\left(q_B\right) = B$ we find after a straightforward calculation that:

$$\pm q_B = \mathbf{i} \qquad \text{and} \qquad \pm\, q_{A,j} = \cos\left(\frac{\pi j}{m}\right) + \sin\left(\frac{\pi j}{m}\right)\mathbf{k}. \tag{4.12}$$

In the following we define $q_{A,j}$ and q_B by the positive sign on the left hand sides of (4.12). Then clearly $q_{A,j} = q_{A,1}^j$ is a $2m$-th unit root.

The *binary dihedral group* $\mathbf{D}_m^* \subset SO(4)$ (via left multiplication on $\mathbb{H} \cong \mathbb{R}^4$) of order $4m$ is now given by:

$$\mathbf{D}_m^* := \left\{\pm q_{A,1}^j, \pm q_{A,1}^j \cdot q_B : j = 0, \dots m-1\right\}.$$

According to the classification result in [19] it gives rise to an infinite class of spherical space forms parametrized by m and different from the lens spaces. Our goal in this section is it to derive an explicit formula for the corresponding spectral zeta function and the zeta-regularized determinant.

The characteristic polynomial P_j of $q_{A,j} = q_{A,1}^j$ acting on $\mathbb{H}$ can be calculated as:

$$P_j(\lambda) = \left[\lambda^2 - 2\cos\left(\frac{\pi j}{m}\right)\lambda + 1\right]^2.$$

Hence $\pm q_{A,1}^j$ have the eigenvalues $\{\pm\lambda_j, \pm\lambda_j, \pm\overline{\lambda_j}, \pm\overline{\lambda_j}\} \subset \mathbb{S}^1$ where λ_j denotes the $2m$-th unit root:

$$\lambda_j := \mathrm{e}^{\sqrt{-1}\frac{\pi j}{m}}$$

and in particular $q_{A,j}$ for $j \neq 0$ is fix point free. From Lemma A.2 and (4.3) one has for $j \neq 0$:

$$C_k\left(q_{A,1}^j, \lambda_j\right) = (k+1)\frac{\lambda_1^{-kj}}{1-\lambda_1^{2j}} - \frac{k+1}{m}\sum_{i=1}^m i\lambda_1^{(2i-2-k)j}$$

because λ_1^{2j} is of order m. Similarly, it holds:

$$C_k\left(-q_{A,1}^j, -\lambda_j\right) = (-1)^k C_k\left(q_{A,1}^j, \lambda_j\right).$$

Summing over $j = 1, \dots m-1$ yields:

$$\sum_{j=1}^{m-1}\left\{C_k\left(q_{A,1}^j, \lambda_j\right) + C_k\left(-q_{A,1}^j, -\lambda_j\right)\right\} = (k+1) \times \begin{cases} m+1-2\sigma_m(k) & k \text{ even} \\ 0 & \text{else} \end{cases}$$

where

$$\sigma_m(k) := \sum_{i=1}^m i \times \begin{cases} 1 & \text{if } 2i-2 \equiv k \bmod 2m \\ 0 & \text{else.} \end{cases}$$

For elements in $\mathbf{D}_m^*$ of the form

$$g_j := q_{A,1}^j \cdot q_B = \cos\left(\frac{\pi j}{m}\right)\mathbf{i} + \sin\left(\frac{\pi j}{m}\right)\mathbf{j}$$

the characteristic polynomial $Q_j(\lambda) = \left[\lambda^2 + 1\right]^2$ is independent from j. In particular, g_j for $0 \le j \le m-1$ has the eigenvalues $\{\sqrt{-1}, \sqrt{-1}, -\sqrt{-1}, -\sqrt{-1}\}$. Moreover,

$$C_k\left(g_j, \sqrt{-1}\right) = \frac{k+1}{2}(-1)^{\frac{-k}{2}}$$

and

$$C_k\left(-g_j, -\sqrt{-1}\right) = C_k\left(g_j, -\sqrt{-1}\right) = (-1)^k\frac{k+1}{2}(-1)^{\frac{-k}{2}}.$$

Hence one obtains:

$$\sum_{j=0}^{m-1}\left\{C_k\left(g_j,\sqrt{-1}\right)+C_k\left(-g_j,-\sqrt{-1}\right)\right\}=(k+1)\times\begin{cases}m(-1)^{-\frac{k}{2}} & k \text{ even}\\ 0 & \text{else.}\end{cases}$$

Finally, by using formula (4.5) it follows for the dimension $\mathbf{m}_k(M_G)$ of eigenspaces:

$$\mathbf{m}_k(M_G)=\frac{\mathbf{m}_k}{4m}+(k+1)n_k$$

where the rational numbers n_k are defined by:

$$n_k:=\frac{1}{2m}\times\begin{cases}m+1+m(-1)^{\frac{k}{2}}-2\sigma_m(k) & k \text{ even}\\ 0 & \text{else.}\end{cases}$$

Note that n_k as a function of k is $4m$-periodic. Hence, writing $k=4m\ell+\nu$ where $\nu=0,\dots 4m-1$ one has with $n_{4m}=n_0$ and a calculation similar to the one above Theorem 4.1:

$$\zeta_{M_G}(s)=\frac{\zeta_{\mathbb{S}^3}(s)}{4m}+\frac{1}{(4m)^{2s}}\sum_{\nu=1}^{4m}n_\nu\left[4mH_{\frac{\nu}{4m},\frac{\nu+2}{4m}}(s-1,s)+H_{\frac{\nu}{4m},\frac{\nu+2}{4m}}(s,s)\right].$$

According to Proposition 2.1 the second term on the right hand side extends to an entire function and by calculating the residue in $s=\frac{3}{2}$ it follows that

$$\sum_{\nu=1}^{4m}n_\nu=0.$$

Theorem 4.3. *The zeta-regularized determinant of M_G is given by:*

$$\log\det\Delta_{M_G}=\frac{\log\det\Delta_{\mathbb{S}^3}}{4m}-\sum_{\nu=1}^{4m}n_\nu A_{4m}(\nu)$$

where $A_{4m}(\nu)\in\mathbb{R}$ in terms of the Hurwitz zeta function is defined in Theorem 4.1.

Remark 4.3. In Corollary 2.1 we gave a general formula for the zeta-regularized determinant of the Laplacian on spherical space forms. The examples we have dealt with more explicitly are limited and even for three dimensional spherical space forms we did not cover all possible cases. The distinct eigenvalues of the Laplacian on M_G are known and the most challenging part is the calculation of the corresponding multiplicities. In this matter our analysis depends on the explicit form of the generating function in Theorem 2.1. However, as was remarked in Section 4.1, in the three dimensional case only three different types of polynomials depending on the eigenvalues of $g\in G\setminus\{I\}$ appear in the calculation of these multiplicities. Therefore, in principle we can determine the zeta-regularized determinants for all three dimensional spherical space forms from a matrix representation of the group G in $SO(4)$ by using formula (2.5).

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Appendix A

In the present appendix we collect useful formulas which are used throughout this paper. Moreover, the non-negative quantities $\beta_q^{(m)}(k)$ and $\gamma_q(k)$ appearing in our above formulas are defined. We start with some relations involving the *Hurwitz zeta-function*.

Lemma A.1. *Let B_n denote the n-th Bernoulli polynomial, then:*

(1) $\zeta(-n,x) = -\frac{B_{n+1}(x)}{n+1}$ *(polynomial of degree $n+1$) and*

$$\frac{\partial^k B_n}{\partial x^k}(x) = \frac{n!}{(n-k)!} B_{n-k}(x).$$

(2) $\zeta'(0,x) = \log\Gamma(x) - \frac{1}{2}\log(2\pi)$ *where $x > 0$.*

(3) *For $k \in \mathbb{N}$ and $s \in \mathbb{C} \setminus \{1\}$ it holds:*

$$\zeta_x^{(k)}(s,x) = \begin{cases} (-1)^k \zeta(s+k,x) \prod_{j=0}^{k-1} (s+j) & \text{if } s \neq 1-k, \\ -(k-1)! & \text{if } s = 1-k. \end{cases}$$

(4) *For $r > 0$ and $t > 0$:*

$$\int_0^{\beta-\alpha} (\beta-\alpha-x)^{r-1} x^{t-1} \mathrm{d}x = (\beta-\alpha)^{r+t-1} \frac{\Gamma(r)\Gamma(t)}{\Gamma(r+t)}.$$

(5) *For $d \in \mathbb{N}_0$ and $\beta > \alpha > 0$:*

$$\zeta(-d,\alpha) - \sum_{l=0}^{d} \binom{d}{l} \zeta(-l,\beta)(\alpha-\beta)^{d-l} = -\frac{(\alpha-\beta)^{d+1}}{d+1}. \tag{A.1}$$

Proof. The identities (1)–(4) are well-known and we only prove (5). According to (1) the assignment $\alpha \mapsto \zeta(-d,\alpha)$ is a polynomial of maximal degree $d+1$. Therefore it coincides with its $(d+1)$-th order Taylor expansion in β. Using (1) one has:

$$\zeta_x^{(l)}(-d,\beta) = -\frac{d!}{(d+1-l)!} B_{d+1-l}(\beta) \tag{A.2}$$

and

$$\begin{aligned} \zeta(-d,\alpha) &= -\sum_{l=0}^{d} \frac{d!}{l!(d+1-l)!} B_{d+1-l}(\beta)(\alpha-\beta)^l - \frac{B_0(\beta)}{d+1}(\alpha-\beta)^{d+1} \\ &= \sum_{l=0}^{d} \binom{d}{l} \left\{ -\frac{B_{l+1}(\beta)}{l+1} \right\} (\alpha-\beta)^{d-l} - \frac{(\alpha-\beta)^{d+1}}{d+1}. \end{aligned}$$

Using (1) again, the identity (A.1) follows. □

Frequently the explicit form of B_n is used. Therefore we list several *Bernoulli polynomials* below. B_n is given as coefficients of the Taylor expansion:

$$\frac{u\mathrm{e}^{xu}}{\mathrm{e}^u - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{u^n}{n!},$$

where $x \in \mathbb{R}$ and $|u| < 2\pi$. In particular,

$$\begin{aligned} B_0(x) &= 1 \\ B_1(x) &= x - \frac{1}{2} \\ B_2(x) &= x^2 - x + \frac{1}{6} \\ B_3(x) &= x(x-1)\left(x - \frac{1}{2}\right). \end{aligned}$$

Lemmas A.2 and A.3 are crucial in calculating the eigenspace dimensions of the Laplacian on spherical space forms. Let $q \in \mathbb{N}$:

Lemma A.2. *Let $\gamma \neq 1$ be a q-th root of unity, then $\sum_{i=1}^{q} i\gamma^i = -q\gamma(1-\gamma)^{-1}$.*

Proof. Use the well-known formula:

$$\sum_{i=1}^{k} ix^i = \frac{kx^{k+2}-(k+1)x^{k+1}+x}{(1-x)^2}. \quad \square$$

For $i := (i_1,\ldots,i_m) \in \mathbb{N}^m$ we write $|i| := i_1+\cdots+i_m$. In the following we use the integers:

$$\beta_q^{(m)}(r) = \sum_{i_1\cdots i_m=1}^{q} i_1\cdots i_m \times \begin{cases} 1 & \text{if } 2m-2|(i_1,\ldots,i_m)| \equiv r \bmod q \\ 0 & \text{else} \end{cases} \tag{A.3}$$

where $r \in \mathbb{Z}$. Note that (A.3) is q-periodic in r. In particular, for fixed $q, m \in \mathbb{N}$ the sequence $(\beta_q^{(m)}(r))_{r\in\mathbb{Z}}$ is bounded.

Lemma A.3. *Let $r \in \mathbb{Z}$ and $q \in \mathbb{N}$ an odd integer. For $p \in \{1,\ldots q-1\}$ with $(p,q)=1$ set $\gamma := \exp(2\pi\sqrt{-1}p/q)$. For $m \in \mathbb{N}$:*

$$\sum_{j=1}^{q-1} \frac{\gamma^{jr}}{(1-\gamma^{2j})^m} = (-1)^m \left[\frac{\beta_q^{(m)}(r)}{q^{m-1}} - \frac{(q+1)^m}{2^m}\right]. \tag{A.4}$$

Proof. We prove (A.4) by induction with respect to m. Assume, that $m=1$ and apply Lemma A.2:

$$\sum_{j=1}^{q-1} \frac{\gamma^{jr}}{1-\gamma^{2j}} = \sum_{j=1}^{q-1} \gamma^{j(r-2)} \frac{\gamma^{2j}}{1-\gamma^{2j}} = -\frac{1}{q}\sum_{i=1}^{q} i \sum_{j=1}^{q-1} \gamma^{j(r-2+2i)}.$$

Note, that

$$\sum_{j=1}^{q-1} \gamma^{j(r-2+2i)} = \begin{cases} q-1 & \text{if } r-2(1-i) \equiv 0 \bmod q, \\ -1 & \text{else} \end{cases}$$

which shows (A.4) in the case $m=1$:

$$\sum_{j=1}^{q-1} \frac{\gamma^{jr}}{1-\gamma^{2j}} = -\beta_q^{(1)}(r) + \frac{1}{q}\sum_{i=1}^{q} i = -\beta_q^{(1)}(r) + \frac{q+1}{2}. \tag{A.5}$$

To show $m \to m+1$ we use Lemma A.2 again:

$$\begin{aligned}\sum_{j=1}^{q-1} \frac{\gamma^{jr}}{(1-\gamma^{2j})^{m+1}} &= -\frac{1}{q}\sum_{i_{m+1}=1}^{q} i_{m+1} \sum_{j=1}^{q-1} \frac{\gamma^{j(r-2+2i_{m+1})}}{(1-\gamma^{2j})^m} \\ &= \frac{(-1)^{m+1}}{q^{m+1}} \sum_{i_{m+1}=1}^{q} i_{m+1}\beta^{(m)}(r-2+2i_{m+1}) + (-1)^m \frac{(q+1)^m}{q2^m} \sum_{i_{m+1}=1}^{q} i_{m+1}.\end{aligned}$$

Finally, notice that

$$\sum_{i_{m+1}=1}^{q} i_{m+1}\beta^{(m)}(r-2+2i_{m+1}) = \beta_q^{(m+1)}(r). \quad \square$$

We derive a second expression for the integers $\beta_q^{(m)}(r)$ in the case where $m=1$ and q is an odd integer. Let us define for $k \in \mathbb{Z}$:

$$\gamma_q(k) := \sharp\{r : k \equiv 2r \bmod q \text{ where } r = 0,\ldots k\}. \tag{A.6}$$

Lemma A.4. *With $k \in \mathbb{N}$ and q odd, $\beta_q^{(1)}(-k)$ and $\gamma_q(k)$ are related via:*

$$\beta_q^{(1)}(-k) = -\frac{q\gamma_q(k)}{2} + \frac{k+q}{2} + 1. \tag{A.7}$$

In particular, there is an estimate: $\gamma_q(k) \le (k+q+2)/q$.

Proof. We calculate the left hand side of (A.4) (which is real due to Lemma A.3) for $m = 1$ in a second way:

$$\sum_{j=1}^{q-1} \frac{\gamma^{-jk}}{1-\gamma^{2j}} = \mathrm{Re} \sum_{j=1}^{q-1} \frac{\gamma^{-j(k+1)}}{\gamma^{-j} - \gamma^{j}} = \frac{1}{2}\sum_{j=1}^{q-1} \gamma^{-jk} \frac{1-\gamma^{2j(k+1)}}{1-\gamma^{2j}}$$
$$= \frac{1}{2}\sum_{r=0}^{k}\sum_{j=1}^{q-1} \gamma^{j(2r-k)} = \frac{1}{2}\left[q\gamma_q(k) - (k+1)\right].$$

According to Lemma A.3 one has:

$$\sum_{j=1}^{q-1} \frac{\gamma^{-jk}}{1-\gamma^{2j}} = -\beta_q^{(1)}(-k) + \frac{q+1}{2}.$$

By comparing these expressions the assertion follows. □

Appendix B

There are many ways to obtain the analytic continuation of the function $H_{\alpha,\beta}(s_1, s_2)$. In this Appendix B we explain a method different from our previous one to express the derivative of the Dirichlet series (1.3) at $s = 0$ in the form of an infinite series in the cases $d = 0, 1$. We use *Weierstrass's canonical form* of the Γ-function for $d = 0$ and we write the analytic extension of (1.3) in terms of *Barne's G-function G* (cf. [17,18]) in the case where $d = 1$. By comparing these formulas with the corresponding ones given above we can derive a relation between G, the Γ-function and the derivative of the Hurwitz zeta function (cf. Proposition B.1). Define

$$\Phi_{(d,\alpha,\beta)}(s) = \sum_{k=1}^{\infty} \frac{1}{(k+\alpha)^{s-d}(k+\beta)^{s}} = H_{\alpha,\beta}(s-d, s) - \frac{1}{\alpha^{s-d}\beta^{s}}.$$

Here we consider the sum for $k > 0$ and we only deal with the cases $d = 0$ and $d = 1$ as examples. The constants α and β are assumed to be positive, but they may take complex values with $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > -1$.

I. $d = 0$.

$$\Phi_{(0,\alpha,\beta)}(s) = \sum_{k=1}^{\infty} \frac{1}{k^{2s}} \frac{1}{(1+\alpha/k)^{s}(1+\beta/k)^{s}} = \sum_{k=1}^{\infty} \frac{1}{k^{2s}} \left(\sum_{n=0}^{\infty} \frac{(\log(1+\alpha/k)(1+\beta/k))^{n}}{n!} (-s)^{n} \right)$$
$$= \sum_{k=1}^{\infty} \frac{1}{k^{2s}} \left(1 - s\log(1+\alpha/k)(1+\beta/k) + s^{2} R_k^{(2)}(\alpha,\beta,s)\right).$$

The functions $R_k^{(2)}(\alpha,\beta,s)$ $(k = 1, 2, \ldots)$ are given by

$$\sum_{n=2}^{\infty} \frac{(\log(1+\alpha/k)(1+\beta/k))^{n}}{n!} (-s)^{n-2}$$

and $R_k^{(2)}(\alpha,\beta,s)$ is holomorphic in s on the whole complex plane for any k.

For $n \ge 2$ the functions

$$\sum_{k=1}^{\infty} \frac{1}{k^{2s+n}}$$

are uniformly bounded on the region $\mathrm{Re}(s) \geq -1/2 + \epsilon$ ($\epsilon > 0$). Hence according to the estimate

$$\left|\sum_{k=1}^{\infty} \frac{1}{k^{2s}} R_k^{(2)}(\alpha, \beta, s)\right| \leq \sum_{n=2}^{\infty} \sum_{k=1}^{\infty} \left|\frac{1}{k^{2s}}\right| \left(\frac{\alpha+\beta}{k}\right)^n \frac{|s|^{n-2}}{n!} \tag{B.1}$$

the function

$$\sum_{k=1}^{\infty} \frac{1}{k^{2s}} R_k^{(2)}(\alpha, \beta, s)$$

is holomorphic on the domain $\mathrm{Re}(s) > -1/2$. Now, the derivative of the function

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{k^{2s}} (1 - s\log(1+\alpha/k)(1+\beta/k)) &= \zeta(2s) - s\sum_{k=1}^{\infty} \frac{1}{k^{2s}} (\log(1+\alpha/k)(1+\beta/k) - (\alpha+\beta)/k) \\ &\quad - s(\alpha+\beta)\zeta(2s+1)\end{aligned}$$

at $s = 0$ coincides with the derivative of the Dirichlet series $\Phi_{(0,\alpha,\beta)}(s)$ at $s = 0$ and

$$\frac{d\Phi_{(0,\alpha,\beta)}(s)}{\mathrm{d}s}_{|s=0} = 2\zeta'(0) - \sum_{k=1}^{\infty} (\log(1+\alpha/k)(1+\beta/k) - (\alpha+\beta)/k) - (\alpha+\beta)\frac{d(s\cdot\zeta(2s+1))}{\mathrm{d}s}_{|s=0}.$$

By using *Weierstrass's canonical form* of the Γ-function, cf. [1]:

$$\frac{1}{\Gamma(s)} = s\cdot \mathrm{e}^{\mathbf{C}s} \prod_{k=1}^{\infty} (1+s/k)\,\mathrm{e}^{-s/k}$$

($\mathbf{C}$ is Euler's constant) we have

$$\begin{aligned}\frac{d\Phi_{(0,\alpha,\beta)}(s)}{\mathrm{d}s}_{|s=0} &= 2\zeta'(0) + \log\Gamma(\alpha+1)\Gamma(\beta+1) + (\alpha+\beta)\left(\mathbf{C} - (s\zeta(2s+1))'_{|s=0}\right) \\ &= -\log 2\pi + \log\Gamma(\alpha+1)\Gamma(\beta+1) + (\alpha+\beta)(\mathbf{C}-\mathbf{C}) \\ &= \log\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{2\pi}\end{aligned}$$

which coincides with our previous results in Proposition 3.3 and Example 3.3.

II. $d = 1$.

We decompose

$$\Phi_{(1,\alpha,\beta)}(s) = \alpha\cdot\Phi_{(0,\alpha,\beta)}(s) + K_{(1,\alpha,\beta)}(s)$$

where the remainder term $K_{(1,\alpha,\beta)}(s)$ is defined by

$$K_{(1,\alpha,\beta)}(s) = \sum_{k=1}^{\infty} \frac{k}{(k+\alpha)^s(k+\beta)^s}.$$

Similar to the case $d = 0$ we have:

$$\begin{aligned}K_{(1,\alpha,\beta)}(s) &= \sum_{k=1}^{\infty} \frac{1}{k^{2s-1}} \frac{1}{(1+\alpha/k)^s(1+\beta/k)^s} = \sum_{k=1}^{\infty} \frac{1}{k^{2s-1}} \Big(1 - s\log(1+\alpha/k)(1+\beta/k) \\ &\quad + \frac{s^2}{2}\left[\log(1+\alpha/k)(1+\beta/k)\right]^2 - s^3 R_k^{(3)}(\alpha,\beta,s)\Big) \\ &= \sum_{k=1}^{\infty} \frac{1}{k^{2s-1}} - s\sum_{k=1}^{\infty} \frac{1}{k^{2s-1}} \left(\log(1+\alpha/k)(1+\beta/k) - \alpha/k + \alpha^2/2k^2 - \beta/k + \beta^2/2k^2\right) \\ &\quad - s\sum_{k=1}^{\infty} \frac{1}{k^{2s-1}} \left(\alpha/k - \alpha^2/2k^2 + \beta/k - \beta^2/2k^2\right)\end{aligned}$$

$$+\frac{s^2}{2!}\sum_{k=1}^{\infty}\frac{1}{k^{2s-1}}\left(\left[\log(1+\alpha/k)(1+\beta/k)\right]^2-\left(\frac{\alpha+\beta}{k}\right)^2\right)$$
$$+\frac{s^2}{2!}(\alpha+\beta)^2\sum_{k=1}^{\infty}\frac{1}{k^{2s+1}}-s^3\sum_{k=1}^{\infty}\frac{1}{k^{2s-1}}R_k^{(3)}(\alpha,\beta,s).$$

By noting that

- $\log(1+\alpha/k)-\alpha/k+\alpha^2/2k^2=O(\alpha^3/k^3)$,
- $\left[\log(1+\alpha/k)\right]^2-(\alpha/k)^2=O(\alpha^3/k^3)$

for $k\to\infty$ together with an estimate on $R_k^{(3)}$ similar to (B.1) one obtains for the derivative of $K_{(1,\alpha,\beta)}(s)$ at $s=0$:

$$\begin{aligned}\frac{\mathrm{d}K_{(1,\alpha,\beta)}(s)}{\mathrm{d}s}_{|s=0} &= 2\zeta'(-1)-(\alpha+\beta)\left(s\zeta(2s)\right)'_{|s=0}+\frac{\alpha^2+\beta^2}{2}\left(s\zeta(2s+1)\right)'_{|s=0}\\ &\quad+\frac{(\alpha+\beta)^2}{2}\left(s^2\zeta(2s+1)\right)'_{|s=0}-\sum_{k=1}^{\infty}k\left(\log(1+\alpha/k)(1+\beta/k)-\frac{\alpha+\beta}{k}+\frac{\alpha^2+\beta^2}{2k^2}\right)\\ &= 2\zeta'(-1)+\frac{\alpha+\beta}{2}+\mathbf{C}\frac{\alpha^2+\beta^2}{2}+\frac{(\alpha+\beta)^2}{4}\\ &\quad-\sum_{k=1}^{\infty}k\left(\log(1+\alpha/k)(1+\beta/k)-\frac{\alpha+\beta}{k}+\frac{\alpha^2+\beta^2}{2k^2}\right).\end{aligned}$$

Recall, that the *Barne G-function* $G(z+1)$ [=the double Γ-function $\Gamma_2(z)$] is defined by:

$$G(z+1)=(2\pi)^{z/2}\mathrm{e}^{-z(z+1)/2-\mathbf{C}z^2/2}\prod_{k=1}^{\infty}\left[\left(1+\frac{z}{k}\right)^k\mathrm{e}^{-z+z^2/2k}\right]. \tag{B.2}$$

After applying logarithms to both sides of (B.2) one obtains:

$$\sum_{k=1}^{\infty}k\left(\log(1+\alpha/k)-\frac{\alpha}{k}+\frac{\alpha^2}{2k^2}\right)=\log G(\alpha+1)-\frac{\alpha}{2}\cdot\log 2\pi+\frac{\alpha(\alpha+1)}{2}+\mathbf{C}\frac{\alpha^2}{2}.$$

Now,

$$\begin{aligned}H'_{\alpha,\beta}(-1,0) &= \alpha\Phi'_{(0,\alpha,\beta)}(0)-\alpha^2\log\alpha\beta+K'_{(1,\alpha,\beta)}(0)\\ &= \alpha\log\frac{\Gamma(\alpha)\Gamma(\beta)}{2\pi}+2\zeta'(-1)+\frac{\alpha+\beta}{2}\log 2\pi-\left(\frac{\alpha-\beta}{2}\right)^2-\log G(\alpha+1)G(\beta+1).\end{aligned}$$

By evaluating this expression for $\alpha=\beta$ [or comparison with Example 3.3, (ii)] we have:

Proposition B.1. *For $\alpha>0$:*

$$\zeta'(-1,\alpha)-\zeta'(-1)=\log\frac{\Gamma(\alpha)^{\alpha}}{G(\alpha+1)}=\log\frac{\Gamma(\alpha)^{\alpha+1}}{G(\alpha)}.$$

Proof. In the second equality we have used $G(\alpha+1)=G(\alpha)/\Gamma(\alpha)$. □

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